Thermodynamics of rotating charged dilaton black holes in an external magnetic field

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Abstract

In the present paper we study the long-standing problem for the thermodynamics of magnetized dilaton black holes. For this purpose we construct an exact solution describing a rotating charged dilaton black hole immersed in an external magnetic field and discuss its basic properties. We derive a Smarr-like relation and the thermodynamics first law for these magnetized black holes. The novelty in the thermodynamics of the magnetized black holes is the appearance of new terms proportional to the magnetic momentum of the black holes in the Smarr-like relation and the first law.

1 Introduction

Black holes influenced by external fields are interesting subject to study and have stimulated a lot of research during the last three decades. Black holes immersed in external magnetic field (the so-called magnetized black holes) are of particular interest to astrophysics where the magnetic field plays important role in the physical processes taking place in the black hole vicinity [1]–[26]. The study of black holes in external magnetic field results in finding interesting astrophysical effects as the charge accretion and the flux expulsion from extreme black holes [1], [3], [10]–[13], [16], [23]. From a theoretical point of view, it is very interesting to investigate how the external fields influence the black hole thermodynamics – more specifically, how the external magnetic field affects it.

The expectation was that a thermodynamic description of the black holes in external magnetic field would include also the value of the background magnetic field as a further parameter. However, more detailed studies showed that this was not the case for static electrically uncharged solutions in four dimensions and some electrically uncharged rotating black holes in higher dimensions – the external magnetic field only distorts the horizon geometry without affecting the thermodynamics [17], [18], [21]. In a recent paper [26] the author studied the thermodynamics of some charged dilaton

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black holes in external magnetic field and he found that even in this case the black hole thermodynamics is not affected by the external magnetic field.

A natural and a long-standing problem is whether this remains true in the rotating case. More or less surprisingly this problem has not been investigated systematically in the literature even for the magnetized Kerr-Newman solution in Einstein-Maxwell gravity. One possible reason for this is the fact that the magnetized Kerr-Newman solution is rather complicated.

The main purpose of the present paper is to study the long-standing problem for the thermodynamics of the magnetized rotating and electrically charged black holes. More precisely we study the thermodynamics of electrically charged and rotating black holes in Einstein-Maxwell-dilaton (EMd) gravity. For this purpose we construct an exact solution describing a rotating charged dilaton black hole with a dilaton coupling parameter $\alpha = \sqrt{3}$ immersed in an external magnetic field and discuss its basic properties. Then we show how the external magnetic field affects the thermodynamics by deriving a Smarr-like relation and the first law. The novelty in the thermodynamics of the magnetized black holes is the appearance of new terms proportional to the magnetic momentum of the black holes in the Smarr-like relation and the first law.

2 Exact rotating charged dilaton black hole in an external magnetic field

The EMd gravity is described by the following field equations

$$R_{\mu\nu} = 2\nabla_\mu\varphi\nabla_\nu\varphi + 2e^{-2\alpha\varphi}\left(F_{\mu\sigma}F_\nu{}^\sigma - \frac{g_{\mu\nu}}{4}F_{\rho\sigma}F^{\rho\sigma}\right), \tag{1}$$

$$\nabla_\mu\left(e^{-2\alpha\varphi}F^{\mu\nu}\right) = 0 = \nabla_{[\mu}F_{\nu\sigma]}, \tag{2}$$

$$\nabla_\mu\nabla^\mu\varphi = -\frac{\alpha}{2}e^{-2\alpha\varphi}F_{\rho\sigma}F^{\rho\sigma}, \tag{3}$$

where ∇_μ and $R_{\mu\nu}$ are the Levi-Civita connection and the Ricci tensor with respect to the spacetime metric $g_{\mu\nu}$. $F_{\mu\nu}$ is the Maxwell tensor and the dilaton field is denoted by φ, with α being the dilaton coupling parameter governing the coupling strength of the dilaton to the electromagnetic field.

The asymptotically flat, rotating and electrically charged dilaton black holes with $\alpha = \sqrt{3}$ are described by the following exact solution to the EMd equations [27]

$$\begin{aligned} ds_0^2 = {} & \left[H(r^2+a^2) + a^2H^{-1}Z\sin^2\theta\right]\sin^2\theta\left(d\phi - \frac{a}{\sqrt{1-\upsilon^2}}\frac{H^{-1}Z}{\left[H(r^2+a^2)+a^2H^{-1}Z\sin^2\theta\right]}dt\right)^2 \\ & - \left(H^{-1}(1-Z) + \frac{a^2}{1-\upsilon^2}\frac{H^{-2}Z^2\sin^2\theta}{\left[H(r^2+a^2)+a^2H^{-1}Z\sin^2\theta\right]}\right)dt^2 + H\frac{\Sigma}{\Delta}dr^2 + H\Sigma d\theta^2, \end{aligned} \tag{4}$$

$$A_t^0 = \frac{\upsilon}{2(1-\upsilon^2)}H^{-2}Z, \tag{5}$$

$$A^0_\phi = -\frac{a\upsilon}{2\sqrt{1-\upsilon^2}} H^{-2} Z \sin^2\theta, \tag{6}$$

$$e^{\frac{2}{3}\sqrt{3}\varphi_0} = H^{-1}, \tag{7}$$

where the functions H, Z, Σ and Δ are given by

$$H = \sqrt{\frac{1-\upsilon^2+\upsilon^2 Z}{1-\upsilon^2}}, \quad Z = \frac{2mr}{\Sigma}, \tag{8}$$

$$\Sigma = r^2 + a^2\cos^2\theta, \quad \Delta = r^2 - 2mr + a^2. \tag{9}$$

The solution parameters υ, m and a are related to the mass M_0, the angular momentum J_0 and the charge Q_0 of the black hole via the formulae

$$M_0 = m\left(1 + \frac{1}{2}\frac{\upsilon^2}{1-\upsilon^2}\right), \quad J_0 = \frac{ma}{\sqrt{1-\upsilon^2}}, \quad Q_0 = \frac{m\upsilon}{1-\upsilon^2}. \tag{10}$$

The event horizon lies at

$$r_+ = m + \sqrt{m^2 - a^2}, \tag{11}$$

which is the greater root of $\Delta = 0$ and the angular velocity of the horizon is

$$\Omega^0_H = \sqrt{1-\upsilon^2}\frac{a}{r_+^2 + a^2}. \tag{12}$$

Another very important characteristic of the black hole, which will play a crucial role in the thermodynamics of the magnetized black holes, is the magnetic momentum μ defined from the asymptotic behavior of A^0_ϕ for $r \to \infty$, namely

$$A^0_\phi \to -\mu\frac{\sin^2\theta}{r} \tag{13}$$

and given by

$$\mu = \frac{ma\upsilon}{\sqrt{1-\upsilon^2}} = \upsilon J_0 = \sqrt{1-\upsilon^2} a Q_0. \tag{14}$$

In order to construct the exact solution describing rotating and charged dilaton black holes in an external magnetic field we shall follow the following scheme. We consider the 4D rotating charged dilaton black holes with a metric

$$ds^2_0 = g^0_{\mu\nu}dx^\mu dx^\nu = X_0(d\phi + W^0 dt)^2 + X_0^{-1}\gamma_{ab}dx^a dx^b, \tag{15}$$

scalar field φ_0 and gauge potential A^0_μ. Its Kaluza-Klein uplifting to a 5D vacuum Einstein solution is given by

$$ds_5^2 = e^{\frac{2}{3}\sqrt{3}\varphi_0} ds_0^2 + e^{-\frac{4}{3}\sqrt{3}\varphi_0}\left(dx_5 + 2A^0_\mu dx^\mu\right)^2. \tag{16}$$

Then we perform a twisted Kaluza-Klein reduction along the Killing field $V = B\frac{\partial}{\partial\phi} + \frac{\partial}{\partial x_5}$, which gives the following 4D solution to the EMd equations with $\alpha = \sqrt{3}$:

$$ds^2 = \Lambda^{-1/2} X_0 (d\phi + W dt)^2 + \Lambda^{1/2} X_0^{-1} \gamma_{ab} dx^a dx^b, \tag{17}$$

$$W = (1 + 2BA^0_\phi) W^0 - 2BA^0_t, \tag{18}$$

$$A_\phi = \Lambda^{-1}(1 + 2BA^0_\phi) A^0_\phi + \frac{B}{2}\Lambda^{-1} X_0 e^{2\sqrt{3}\varphi_0}, \tag{19}$$

$$A_t = \Lambda^{-1}(1 + 2BA^0_\phi) A^0_t + \frac{B}{2}\Lambda^{-1} X_0 e^{2\sqrt{3}\varphi_0} W^0, \tag{20}$$

$$e^{\frac{4}{3}\sqrt{3}\varphi} = \Lambda^{-1} e^{\frac{4}{3}\sqrt{3}\varphi_0}, \tag{21}$$

where Λ is given by

$$\Lambda = \left(1 + 2BA^0_\phi\right)^2 + B^2 X_0 e^{2\sqrt{3}\varphi_0}. \tag{22}$$

Alternative derivation of the rotating and electrically charged dilaton black hole solution is given in the appendix.

We start investigating the basic properties of the magnetized solution by noting that there is no conical singularity on the axis of symmetry since both A^0_ϕ and X_0 vanish at the axis. This means that the periodicity of the angular coordinate ϕ is the standard one $\Delta\phi = 2\pi$. As in the non-rotating case the parameter B is the asymptotic strength of the magnetic field. The inspection of the metric shows that there is a Killing horizon at $r = m + \sqrt{m^2 - a^2}$ where the Killing field $K = \frac{\partial}{\partial t} + \Omega_H \frac{\partial}{\partial\phi}$ becomes null. Here Ω_H is the angular velocity of the horizon given by

$$\begin{gathered}\Omega_H = -W|_H = -W^0|_H - 2B(A^0_t - A^0_\phi W^0)_H = \\ \Omega^0_H - 2B(A^0_t + \Omega^0_H A^0_\phi)|_H = \Omega^0_H - 2B\Xi^0_H = \Omega^0_H - B\upsilon,\end{gathered} \tag{23}$$

where Ω^0_H is the horizon angular velocity and $\Xi^0_H = (A^0_t + \Omega^0_H A^0_\phi)|_H = \frac{1}{2}\upsilon$ is the corotating electric potential evaluated on the horizon for the non-magnetized seed solution.

The metric induced on the horizon cross section is

$$ds_H^2 = \Lambda^{-1/2} X_0 d\phi^2 + \Lambda^{1/2} g^{(0)}_{\theta\theta} d\theta^2 =$$
$$\Lambda^{-1/2} \left[H(r^2+a^2) + a^2 H^{-1} Z \sin^2\theta \right] \sin^2\theta d\phi^2 + \Lambda^{1/2} H \Sigma d\theta^2. \tag{24}$$

The horizon area calculated via (24) coincides with the horizon area of the non-magnetized seed solution and is given by

$$\mathcal{A}_H = \mathcal{A}^0_H = \frac{4\pi(r_+^2+a^2)}{\sqrt{1-\upsilon^2}}. \tag{25}$$

Although the horizon cross section is a topological 2-sphere, it is geometrically deformed by the rotation and the external magnetic field. The horizon surface gravity κ can be found by the well-known formula

$$\kappa^2 = -\frac{1}{4\lambda} g^{\mu\nu} \partial_\mu \lambda \partial_\nu \lambda, \tag{26}$$

where $\lambda = g(K, K)$. By direct computation we find that the surface gravity of the magnetized solution coincides with the surface gravity of the seed solution

$$\kappa = \kappa_0 = \sqrt{1-\upsilon^2}\, \frac{r_+ - r_-}{(r_+^2+a^2)} = \sqrt{1-\upsilon^2}\, \frac{\sqrt{m^2-a^2}}{(r_+^2+a^2)}. \tag{27}$$

The solution possesses an ergoregion determined by the inequality $g(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}) = g_{tt} > 0$. The explicit form of g_{tt} can be presented as follows

$$g_{tt} = \sqrt{\Lambda} \left[g^0_{tt} - X_0 (W^0)^2 + \frac{X_0}{\Lambda} W^2 \right]. \tag{28}$$

Using this form of g_{tt}, it is not difficult to see that very close to the horizon we have $g_{tt} \geq 0$ and $g_{tt} < 0$ for large r. Therefore, we conclude that there is an ergoregion confined in a compact neighborhood of the horizon, in contrast with the magnetized Kerr-Newman solution for which the ergoregion extends to infinity [24].

Concerning the asymptotic of the magnetized solution, it is not difficult to see from its explicit form that the solution asymptotes the static dilaton-Melvin solution with $\alpha = \sqrt{3}$. This is also confirmed by inspecting the asymptotic behavior of the electric field $\vec{E}$ and the twists $\omega_\eta = i_\eta \star d\eta$ and $\omega_\xi = i_\xi \star d\xi$ associated with the Killing vectors $\eta = \frac{\partial}{\partial \phi}$ and $\xi = \frac{\partial}{\partial t}$. For these quantities we have $\vec{E}^2 \to 0$, $\omega_\eta \to 0$ and $\omega_\xi \to 0$ for $r \to \infty$.

3 Thermodynamics

The main purpose of the present paper is to study the thermodynamics of the magnetized rotating, charged dilaton black holes. The first step in this direction is to find

the conserved physical quantities, namely the electric charge, the mass and the angular momentum associated with the magnetized black hole. The physical electric charge can be calculated as usual via the well-known formula and the result is

$$Q = \frac{1}{4\pi}\int_{H} e^{-2\sqrt{3}\varphi} \star F = Q_0 + 2BJ_0. \tag{29}$$

The calculation of the mass and the angular momentum, however, is much more tricky since the spacetime is not asymptotically flat – a substraction procedure is needed to obtain finite quantities from integrals divergent at infinity. The natural choice for the substraction background in our case is the static dilaton-Melvin background. To calculate the mass we use the quasilocal formalism [23]. Here we give for completeness a very brief description of the quasilocal formalism. We foliate the spacetime by spacelike surfaces Σ_t of metric $h_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu$, labeled by a time coordinate t with a unit normal vector $u^\mu = -N\delta^\mu_0$. The spacetime metric is then decomposed into the form

$$ds^2 = -N^2dt^2 + h_{ij}(dx^i + N^i dt)(dx^j + N^j dt), \tag{30}$$

with N and N^i being the lapse function and the shift vector.

The spacetime boundary consists of the initial surface Σ_i $(t = t_i)$, the final surface Σ_f $(t = t_f)$ and a timelike surface $\mathcal{B}$ to which the vector u^μ is tangent. The surface $\mathcal{B}$ is foliated by 2-dimensional surfaces S^r_t, with metric $\sigma_{\mu\nu} = h_{\mu\nu} - n_\mu n_\nu$, which are the intersections of Σ_t and $\mathcal{B}$. The unit spacelike outward normal to S^r_t, n_μ, is orthogonal to u^μ.

Let us denote by K the trace of the extrinsic curvature $K^{\mu\nu}$ of $\Sigma_{t_{i,f}}$ and by Θ the trace of the extrinsic curvature $\Theta^{\mu\nu}$ of $\mathcal{B}$, given by

$$\begin{aligned} K_{\mu\nu} &= -\frac{1}{2N}\left(\frac{\partial h_{\mu\nu}}{\partial t} - 2D_{(\mu}N_{\nu)}\right), && (31)\\ \Theta_{\mu\nu} &= -h^\alpha_\mu \nabla_\alpha n_\nu, && (32) \end{aligned}$$

where ∇_μ and D_ν are the covariant derivatives with respect to the metric $g_{\mu\nu}$ and $h_{\mu\nu}$, respectively. The quasilocal energy M and the angular momentum J_i are given by

$$\begin{aligned} M = \frac{1}{8\pi}\int_{S^r_t}\sqrt{\sigma}\left[N(k-k_*) + \frac{n_\mu p^{\mu\nu}N_\nu}{\sqrt{h}}\right] d^{D-2}x \\ +\frac{1}{4\pi}\int_{S^r_t} A_t\left(\hat{\Pi}^j - \hat{\Pi}^j_*\right) n_j d^{D-2}x, && (33) \end{aligned}$$

$$J_i = -\frac{1}{8\pi}\int_{S^r_t}\frac{n_\mu p^\mu_i}{\sqrt{h}}\sqrt{\sigma}d^{D-2}x - \frac{1}{4\pi}\int_{S^r_t} A_i\hat{\Pi}^j n_j d^{D-2}x. \tag{34}$$

Here $k = -\sigma^{\mu\nu}D_\nu n_\mu$ is the trace of the extrinsic curvature of S^r_t embedded in Σ_t. The momentum variable p^{ij} conjugated to h_{ij} is given by

$$p^{ij} = \sqrt{h}\left(h^{ij}K - K^{ij}\right). \tag{35}$$

The quantity $\hat{\Pi}^j$ is defined by

$$\hat{\Pi}^j = -\frac{\sqrt{\sigma}}{\sqrt{h}}\sqrt{-g}e^{-2\alpha\varphi}F^{tj}. \tag{36}$$

The quantities with the subscript "$_*$" are those associated with the background.

Using the above formulae for the quasilocal mass and the quasilocal angular momentum, after rather long but straightforward calculations we find the following result for the total mass and the total angular momentum

$$M = M_0, \quad J = J_0. \tag{37}$$

This result, combined with (23) and (29), gives the following Smarr-like relation for the magnetized black hole

$$M = \frac{1}{4\pi}\kappa\mathcal{A}_H + 2\Omega_H J + \Xi_H Q + \mu B, \tag{38}$$

where $\mu = 2\Xi^0_H J_0$ is the magnetic moment of the non-magnetized black hole defined by (13) and (14), and $\Xi_H = K^\mu A_\mu|_H$ is the corotating electric potential of the magnetized black hole evaluated on the horizon. It is not difficult one to show that $\Xi_H = \Xi^0_H$, which gives $\mu = 2\Xi^0_H J_0 = 2\Xi_H J$.

For the first law of the magnetized black holes we obtain the following expression

$$\delta M = \frac{\kappa}{2\pi}\delta\left(\frac{\mathcal{A}_H}{4}\right) + \Omega_H \delta J + \Xi_H \delta Q - \mu\delta B. \tag{39}$$

The first law can be easily checked by taking into account the fact that $\kappa = \kappa_0$ and $\mathcal{A}_H = \mathcal{A}^0_H$, the explicit expressions for Q and Ω_H, as well as the first law for the non-magnetized black holes $\delta M_0 = \frac{\kappa_0}{2\pi}\delta\left(\frac{\mathcal{A}^0_H}{4}\right) + \Omega^0_H \delta J_0 + \Xi^0_H \delta Q_0$.

From the explicit form of the Smarr-like relation (38) and the first law (39) we see that the thermodynamics of the magnetized rotating electrically charged dilaton black holes indeed includes the background magnetic field as a further parameter conjugated to the magnetic moment of the black holes. The way in which the external magnetic field enters the Smarr-like relation and the first law is what should be expected from a physical point of view. The additional terms associated with the background magnetic field in the Smarr-like relation and the first law are in fact related to the energy of the black hole considered as a magnetic dipole in external magnetic field. This also explains the fact why the static, neutral and electrically charged, magnetized black holes have thermodynamic unaffected by the external field. In order for the background magnetic field to affect the black hole thermodynamics, the magnetic moment has to be non-zero and this occurs only for black holes which are simultaneously electrically charged and with non-zero angular momentum according to the formula $\mu = 2\Xi^0_H J_0 = 2\Xi_H J$.

4 Discussion

In the present paper we derived the Smarr-like relation and the first law for the magnetized rotating and electrically charged dilaton black holes with dilaton coupling parameter $\alpha = \sqrt{3}$. Taking into account the discussion at the end of the previous section, on a physical background one should expect that the Smarr-like relation and the first law for magnetized rotating and electrically charged black holes with an arbitrary dilaton coupling parameter α (including the Kerr-Newman solution corresponding to $\alpha = 0$) would have the same form as in the case $\alpha = \sqrt{3}$. The systematic derivation, however, seems to be nontrivial because in the general case the magnetized solutions would not asymptote to the static dilaton-Melvin solution, as it was demonstrated for the magnetized Kerr-Newman solution in [5] and [24]. In other words, in order to obtain meaningful expressions for the mass and the angular momentum of the magnetized black hole solutions, the substraction procedure has to be carefully chosen in order to cope with the complicated asymptotic behavior of the solutions. In fact there are some results about the first law for the magnetized Kerr-Newman solution given in [28]. However, the mass and angular momentum for the magnetized Kerr-Newman black hole were chosen ad hock in [28] without a systematic derivation. We intend to discussed these problems and their solution in a future publication.

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A Alternative derivation of the magnetized rotating charged dilaton black hole solution with $\alpha = \sqrt{3}$

We consider stationary and axisymmetric EMd spacetimes, i.e spacetimes admitting a Killing field ξ which is timelike at infinity, and a spacelike axial Killing field η with closed orbits. The dimensional reduction of the field equations along the axial Killing field η was performed in [29]. So we present here only the basic steps skipping the details which can be found in [29]. The invariance of the Maxwell 2-form F under the flow of the Killing field η implies the existence of the potentials Φ and Ψ defined by $d\Phi = i_\eta F$ and $d\Psi = -e^{-2\alpha\varphi} i_\eta \star F$ and such that

$$F = X^{-1}\eta \wedge d\Phi - X^{-1}e^{2\alpha\varphi} \star (d\Psi \wedge \eta)\,, \tag{40}$$

where

$$X = g(\eta, \eta). \tag{41}$$

The twist ω_η of the Killing field η, defined by $\omega_\eta = \star(d\eta \wedge \eta)$, satisfies the equation

$$d\omega_\eta = 4d\Psi \wedge d\Phi = d(2\Psi d\Phi - 2\Phi d\Psi). \tag{42}$$

Hence we conclude that there exists a twist potential χ such that $\omega_\eta = d\chi + 2\Psi d\Phi - 2\Phi d\Psi$.

The projection metric γ orthogonal to the Killing field η, is defined by

$$g = X^{-1}\left(\gamma + \eta \otimes \eta\right). \tag{43}$$

In local coordinates adapted to the Killing field, i.e. $\eta = \partial/\partial\phi$, we have

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = X(d\phi + W_a dx^a)^2 + X^{-1}\gamma_{ab}dx^a dx^b. \tag{44}$$

The 1-form $\mathcal{W} = W_a dx^a$ is closely related to the twist ω which can be expressed in the form

$$\omega_\eta = X i_\eta \star d\mathcal{W}. \tag{45}$$

The dimensionally reduced EMd equations for $\alpha = \sqrt{3}$ form an effective 3-dimensional gravity coupled to a nonlinear matrix $SL(3,R)/O(3)$ σ-model with the following action

$$\mathcal{A} = \int d^3x\sqrt{-\gamma}\left[R(\gamma) - \frac{1}{4}\gamma^{ab}Tr\left(S^{-1}\partial_a S S^{-1}\partial_b S\right)\right], \tag{46}$$

where $R(\gamma)$ is the Ricci scalar curvature with respect to the metric γ_{ab} and S is a symmetric $SL(3,R)$ matrix explicitly given by [30]

$$S = e^{-\frac{2\sqrt{3}}{3}\varphi}\begin{pmatrix} X^{-1} & -2X^{-1}\Phi & X^{-1}(2\Phi\Psi - \chi) \\ -2X^{-1}\Phi & e^{2\sqrt{3}\varphi} + 4X^{-1}\Phi^2 & -2e^{2\sqrt{3}\varphi}\Psi - 2X^{-1}(2\Phi\Psi - \chi)\Phi \\ X^{-1}(2\Phi\Psi - \chi) & -2e^{2\sqrt{3}\varphi}\Psi - 2X^{-1}(2\Phi\Psi - \chi)\Phi & X + 4\Psi^2 e^{2\sqrt{3}\varphi} + X^{-1}(2\Phi\Psi - \chi)^2 \end{pmatrix}.$$

The group of symmetries $SL(3,R)$ can be used to generate new solutions from known ones via the scheme

$$S \to \Gamma S \Gamma^T, \quad \gamma_{ab} \to \gamma_{ab}, \tag{47}$$

where $\Gamma \in SL(3,R)$. In the present paper we consider seed solutions with potentials $(X_0, \Phi_0, \Psi_0, \chi_0, \varphi_0)$ corresponding to the seed matrix S_0 and transformation matrices in the form

$$\Gamma = \left(\begin{array}{ccc} 1 & B & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right), \tag{48}$$

with B being an arbitrary real number.

New solutions which can be generated from the seed and the transformation matrices under consideration, are encoded in the matrix $S = \Gamma S_0 \Gamma^T$ and the explicit form of their potentials is given by

$$\begin{aligned}
X &= \frac{X_0}{\sqrt{\Lambda}}, && (49)\\
\Phi &= \Lambda^{-1}\Phi_0(1-2B\Phi_0) - \frac{B}{2}\Lambda^{-1}e^{2\sqrt{3}\varphi_0}X_0, && (50)\\
\Psi &= (1-B\Phi_0)\Psi_0 + \frac{B}{2}\chi_0, && (51)\\
\chi &= \Lambda^{-1}\Big\{(1-2B\Phi_0)\chi_0 + Be^{2\sqrt{3}\varphi_0}X_0\Psi_0 \\
&+ \left[B\Phi_0(1-2B\Phi_0) - \frac{1}{2}B^2e^{2\sqrt{3}\varphi_0}X_0\right](\chi_0 - 2\Phi_0\Psi_0)\Big\}, && (52)\\
e^{\frac{4}{\sqrt{3}}\varphi} &= \Lambda^{-1}e^{\frac{4}{\sqrt{3}}\varphi_0}, && (53)
\end{aligned}$$

where $\Lambda = (1-2B\Phi_0)^2 + B^2X_0e^{2\sqrt{3}\varphi_0}$.

In order to generate the solution describing the magnetized rotating and electrically charged dilaton black holes we have to choose the seed solution to be the asymptotically flat rotating electrically charged dilaton black hole solution with $\alpha = \sqrt{3}$ found in [27].

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